

Idempotent Mathematics and Interval Analysis *

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Abstract

A brief introduction into Idempotent Mathematics and an idempotent version of Interval Analysis are presented. Some applications are discussed.

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1 Introduction

Many problems in the optimization theory and other fields of mathematics appear to be linear over semirings with idempotent addition (the so-called *idempotent superposition principle* [1], which is a natural analog of the well-known superposition principle in Quantum Mechanics). The corresponding approach is developed systematically as *Idempotent Mathematics* or *Idempotent Analysis*, a branch of mathematics which has been growing vigorously last time (see, e.g., [1]–[8]).

One of the most important examples of idempotent semirings is the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equipped with operations $\oplus = \max$, $\odot = +$ (see section 2). In general, there exists a correspondence between interesting, useful, and important constructions and results concerning the field of real (or complex) numbers and similar constructions dealing with various idempotent semirings. This correspondence can be formulated in the spirit of the well known N. Bohr's correspondence principle in Quantum Mechanics; in fact, the two principles are intimately connected (see [4, 5, 6] and sections 5, 6 and 7 below). We discuss idempotent analogs of some basic ideas, constructions, and results in traditional calculus and functional analysis; also, we show that the correspondence principle is a powerful heuristic tool to apply unexpected analogies and ideas borrowed from different areas of mathematics (see, e.g., [1]–[6]).

The theory is well advanced and includes, in particular, new integration theory, new linear algebra, spectral theory, and functional analysis. Its applications include

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various optimization problems such as multicriteria decision making, optimization on graphs, discrete optimization with a large parameter (asymptotic problems), optimal design of computer systems and computer media, optimal organization of parallel data processing, dynamic programming, applications to differential equations, numerical analysis, discrete event systems, computer science, discrete mathematics, mathematical logic, etc. (see, e.g., [2]–[9] and references therein).

In section 2 we give a short heuristic introduction into Idempotent Mathematics. Section 3 contains definitions of basic concepts of idempotent arithmetic and several important examples. In sections 4–7 we consider the notion of linearity in Idempotent Analysis and indicate some of its applications to idempotent linear algebra.

Due to imprecision of sources of input data in real-world problems, the data usually come in a form of confidence intervals or other number sets rather than exact quantities. Interval Analysis (see, e.g., [10]–[13]) extends operations of traditional calculus from numbers to number intervals, thus allowing to process such imprecise data and control rounding errors in computations. To construct the analog of calculus of intervals in the context of optimization theory and Idempotent Analysis, we develop a set-valued extension of idempotent arithmetic (see section 8).

The interval extension of an idempotent semiring is constructed in sections 9 and 10. The idempotent interval arithmetic appears to be remarkably simpler than its traditional analog. For example, in the traditional interval arithmetic multiplication of intervals is not distributive with respect to interval addition, while idempotent interval arithmetic conserves distributivity.

A simple application of interval arithmetic to idempotent linear algebra is discussed in section 12. We stress that in the traditional Interval Analysis the set of all square interval matrices of a given order does not form a semigroup with respect to matrix multiplication: this operation is not associative since distributivity is lost in traditional interval arithmetic. On the contrary, in the idempotent case associativity is conserved.

Two properties that make the idempotent interval arithmetic so simple are monotonicity of arithmetic operations and positivity of all elements of an idempotent semiring. In general, idempotent interval analysis appears to be best suited for treating the problems with order-preserving transformations of imprecise data. We stress that this construction provides another example of heuristic power of the idempotent correspondence principle.

2 Dequantization and idempotent correspondence principle

Let \mathbb{R} be the field of real numbers and \mathbb{R}_+ be the subset of all non-negative numbers. Consider the following change of variable:

$$u \mapsto w = h \ln u,$$

where $u \in \mathbb{R}_+ \setminus \{0\}$, $h > 0$; thus $u = e^{w/h}$, $w \in \mathbb{R}$. Denote by $\mathbf{0}$ the additional element $-\infty$ and by S the extended real line $\mathbb{R} \cup \{\mathbf{0}\}$. The above change of variable has a natural extension D_h to the whole S by $D_h(0) = \mathbf{0}$; also, we denote $D_h(1) = 0 = \mathbf{1}$.

Denote by S_h the set S equipped with the two operations \oplus_h (generalized addition) and \odot_h (generalized multiplication) such that D_h is a homomorphism of $\{\mathbb{R}_+, +, \cdot\}$ to $\{S, \oplus_h, \odot_h\}$. This means that $D_h(u_1 + u_2) = D_h(u_1) \oplus_h D_h(u_2)$ and $D_h(u_1 \cdot u_2) = D_h(u_1) \odot_h D_h(u_2)$, i.e., $w_1 \odot_h w_2 = w_1 + w_2$ and $w_1 \oplus_h w_2 = h \ln(e^{w_1/h} + e^{w_2/h})$. It is easy to prove that $w_1 \oplus_h w_2 \rightarrow \max\{w_1, w_2\}$ as $h \rightarrow 0$.

Denote by \mathbb{R}_{\max} the set $S = \mathbb{R} \cup \{\mathbf{0}\}$ equipped with operations $\oplus = \max$ and $\odot = +$, where $\mathbf{0} = -\infty$, $\mathbf{1} = 0$ as above. Algebraic structures in \mathbb{R}_+ and S_h are isomorphic; therefore \mathbb{R}_{\max} is a result of a deformation of the structure in \mathbb{R}_+ .

We stress the obvious analogy with the quantization procedure, where h is the analog of the Planck constant. In these terms, \mathbb{R}_+ (or \mathbb{R}) plays the part of a ‘quantum object’ while \mathbb{R}_{\max} acts as a ‘classical’ or ‘semiclassical’ object that arises as the result of a *dequantization* of this quantum object.

Likewise, denote by \mathbb{R}_{\min} the set $\mathbb{R} \cup \{\mathbf{0}\}$ equipped with operations $\oplus = \min$ and $\odot = +$, where $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. Clearly, the corresponding dequantization procedure is generated by the change of variables $u \mapsto w = -h \ln u$.

Consider also the set $\mathbb{R} \cup \{\mathbf{0}, \mathbf{1}\}$, where $\mathbf{0} = -\infty$, $\mathbf{1} = +\infty$, together with the operations $\oplus = \max$ and $\odot = \min$. Obviously, it can be obtained as a result of a ‘second dequantization’ of \mathbb{R} or \mathbb{R}_+ .

The algebras presented in this section are the most important examples of idempotent semirings (for the general definition see section 3), the central algebraic structure of Idempotent Analysis. The basic object of the traditional calculus is a *function* defined on some set X and taking its values in the field \mathbb{R} (or \mathbb{C}); its idempotent analog is a map $X \rightarrow S$, where X is some set and $S = \mathbb{R}_{\min}$, \mathbb{R}_{\max} , or another idempotent *semiring*. Let us show that redefinition of basic constructions of traditional calculus in terms of Idempotent Mathematics can yield interesting and nontrivial results (see, e.g., [1]–[7] for details and generalizations).

EXAMPLE 2.1. INTEGRATION AND MEASURES. To define an idempotent analog of the Riemann integral, consider a Riemann sum for a function $\varphi(x)$, $x \in X = [a, b]$, and substitute semiring operations \oplus and \odot for traditional addition and multiplication of real numbers in its expression (for the sake of being definite, consider the semiring \mathbb{R}_{\max}):

$$\sum_{i=1}^N \varphi(x_i) \cdot \Delta_i \quad \mapsto \quad \bigoplus_{i=1}^N \varphi(x_i) \odot \Delta_i = \max_{i=1, \dots, N} (\varphi(x_i) + \Delta_i),$$

where $a = x_0 < x_1 < \dots < x_N = b$, $\Delta_i = x_i - x_{i-1}$, $i = 1, \dots, N$. As $\max_i \Delta_i \rightarrow 0$, the integral sum tends to

$$\int_X^{\oplus} \varphi(x) dx = \sup_{x \in X} \varphi(x)$$

for any function $\varphi: X \rightarrow \mathbb{R}_{\max}$ that is bounded. In general, the set function

$$m_\varphi(B) = \sup_{x \in B} \varphi(x), \quad B \subset X,$$

is called an \mathbb{R}_{\max} -measure on X ; since $m_\varphi(\bigcup_\alpha B_\alpha) = \sup_\alpha m_\varphi(B_\alpha)$, this measure is completely additive. An idempotent integral with respect to this measure is defined as

$$\int_X^\oplus \psi(x) dm_\varphi = \int_X^\oplus \psi(x) \odot \varphi(x) dx = \sup_{x \in X} (\psi(x) + \varphi(x)).$$

EXAMPLE 2.2. FOURIER–LEGENDRE TRANSFORM. Consider the topological group $G = \mathbb{R}^n$. The usual Fourier–Laplace transform is defined as

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) dx,$$

where $\exp(i\xi \cdot x)$ is a *character* of the group G , i.e., a solution of the following functional equation:

$$f(x + y) = f(x)f(y).$$

The idempotent analog of this equation is

$$f(x + y) = f(x) \odot f(y) = f(x) + f(y).$$

Hence ‘idempotent characters’ of the group G are linear functions of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. Thus the Fourier–Laplace transform turns into

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^\oplus \xi \cdot x \odot \varphi(x) dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)).$$

This equation differs from the well-known Legendre–Fenchel transform (see, e.g., [14]) in insignificant details.

These examples suggest the following formulation of the idempotent correspondence principle [4]:

There exists a heuristic correspondence between interesting, useful, and important constructions and results over the field of real (or complex) numbers and similar constructions and results over idempotent semirings in the spirit of N. Bohr’s correspondence principle in Quantum Mechanics.

So Idempotent Mathematics can be treated as a ‘classical shadow (or counterpart)’ of the traditional Mathematics over fields.

3 Idempotent semirings: Basic definitions

Consider a set S equipped with two algebraic operations: *addition* \oplus and *multiplication* \odot . The triple $\{S, \oplus, \odot\}$ is an *idempotent semiring* if it satisfies the following conditions (here and below, the symbol \star denotes any of the two operations \oplus, \odot):

- the addition \oplus and the multiplication \odot are *associative*: $x \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in S$;
- the addition \oplus is *commutative*: $x \oplus y = y \oplus x$ for all $x, y \in S$;
- the addition \oplus is *idempotent*: $x \oplus x = x$ for all $x \in S$;
- the multiplication \odot is *distributive* with respect to the addition \oplus : $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in S$.

In the rest of this paper we shall sometimes drop the word ‘idempotent’ when the context is clear.

A *unity* of an idempotent semiring S is an element $\mathbf{1} \in S$ such that for all $x \in S$

$$\mathbf{1} \odot x = x \odot \mathbf{1} = x.$$

A *zero* of an idempotent semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and for all $x \in S$

$$\mathbf{0} \oplus x = x, \quad \mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}.$$

It is readily seen that if an idempotent semiring S contains a unity (a zero), then this unity (zero) is determined uniquely.

A semiring S is said to be *commutative* if $x \odot y = y \odot x$ for all $x, y \in S$. A commutative semiring is called a *semifield* if every nonzero element of this semiring is invertible. It is clear that \mathbb{R}_{\max} and \mathbb{R}_{\min} are semifields.

Note that different versions of this axiomatics are used, see, e.g., [2]–[8] and some literature indicated in these books and papers.

The addition \oplus defines on an idempotent semiring S a *partial order*: $x \preceq y$ iff $x \oplus y = y$. We use the notation $x \prec y$ if $x \preceq y$ and $x \neq y$. If S contains a zero $\mathbf{0}$, then $\mathbf{0}$ is its least element with respect to the order \preceq . The operations \oplus and \odot are consistent with the order \preceq in the following sense: if $x \preceq y$, then $x \star z \preceq y \star z$ and $z \star x \preceq z \star y$ for all $x, y, z \in S$.

An idempotent semiring S is said to be *a-complete* if for any subset $\{x_\alpha\} \subset S$, including \emptyset , there exists a sum $\bigoplus \{x_\alpha\} = \bigoplus_\alpha x_\alpha$ such that $(\bigoplus_\alpha x_\alpha) \odot y = \bigoplus_\alpha (x_\alpha \odot y)$ and $y \odot (\bigoplus_\alpha x_\alpha) = \bigoplus_\alpha (y \odot x_\alpha)$ for any $y \in S$. An idempotent semiring S containing a zero $\mathbf{0}$ is said to be *b-complete* if the conditions of *a-completeness* are satisfied for any nonempty subset $\{x_\alpha\} \subset S$ that is bounded from above. Any *b-complete* semiring either is *a-complete* or becomes *a-complete* if the greatest element $\infty = \sup S$ is added; see [5, 6] for details.

Note that $\bigoplus_\alpha x_\alpha = \sup \{x_\alpha\}$; in particular, an *a-complete* idempotent semiring always contains the zero $\mathbf{0} = \bigoplus \emptyset$.

An idempotent semiring S does not contain *zero divisors* if $x \odot y = \mathbf{0}$ implies that $x = \mathbf{0}$ or $y = \mathbf{0}$ for all $x, y \in S$. An idempotent semiring S is said to satisfy the *cancellation condition* if for all $x, y, z \in S$ such that $x \neq \mathbf{0}$ it follows from $x \odot y = x \odot z$ or $y \odot x = z \odot x$ that $y = z$. Any idempotent semiring satisfying the cancellation condition does not contain zero divisors. A commutative idempotent

semiring S is said to be *idempotent semifield* if every nonzero element of S is invertible; in this case the cancellation condition is fulfilled.

An idempotent semiring S is said to be *algebraically closed* if the equation $x^n = y$, where $x^n = x \odot x \odot \cdots \odot x$ (n times), has a solution for all $y \in S$ and $n \in \mathbb{N}$; an idempotent semiring S with a unity $\mathbf{1}$ satisfies the *stabilization condition* if the sequence $x^n \oplus y$ stabilizes whenever $x \preceq \mathbf{1}$ and $y \neq \mathbf{0}$ [15, 16]. Note that in [16] the property of algebraic closedness was incorrectly called ‘algebraic completeness’ due to translator’s mistake.

The most important examples of idempotent semirings are those considered in section 2. We see that \mathbb{R}_{\max} is a b -complete algebraically closed idempotent semifield satisfying stabilization condition. The idempotent semiring \mathbb{R}_{\min} is isomorphic to \mathbb{R}_{\max} . Note that both \mathbb{R}_{\max} and \mathbb{R}_{\min} are linearly ordered with respect to the corresponding addition operations; the order \preceq in \mathbb{R}_{\max} coincides with the usual linear order \leq and is opposite to the order \preceq in \mathbb{R}_{\min} .

Consider the set $\widehat{\mathbb{R}}_{\max} = \mathbb{R}_{\max} \cup \{\infty\}$ with operations \oplus, \odot extended by $\infty \oplus x = \infty$ for all $x \in \mathbb{R}_{\max}$, $\infty \odot x = \infty$ if $x \neq \mathbf{0}$ and $\infty \odot \mathbf{0} = \mathbf{0}$. It is easily shown that this set is an a -complete idempotent semiring and ∞ is its greatest element.

Let $\{S_1, S_2, \dots\}$ be a collection of idempotent semirings. There are several ways to construct a new idempotent semiring derived from the semirings of this collection.

EXAMPLE 3.1. Suppose S is an idempotent semiring and X is an arbitrary set. The set $\mathcal{M}(X; S)$ of all functions $X \rightarrow S$ is an idempotent semiring with respect to the following operations:

$$(f \oplus g)(x) = f(x) \oplus g(x), \quad (f \odot g)(x) = f(x) \odot g(x), \quad x \in X.$$

If S contains a zero $\mathbf{0}$ and/or a unity $\mathbf{1}$, then the functions $o(x) = \mathbf{0}$ for all $x \in X$, $e(x) = \mathbf{1}$ for all $x \in X$ are the zero and the unity of the idempotent semiring $\mathcal{M}(X; S)$. It is also possible to consider various subsemirings of $\mathcal{M}(X; S)$.

EXAMPLE 3.2. Let S_i be idempotent semirings with operations \oplus_i, \odot_i and zeros $\mathbf{0}_i$, $i = 1, \dots, n$. The set $S = (S_1 \setminus \{\mathbf{0}_1\}) \times \cdots \times (S_n \setminus \{\mathbf{0}_n\}) \cup \mathbf{0}$ is an idempotent semiring with respect to the following operations: $x \star y = (x_1, \dots, x_n) \star (y_1, \dots, y_n) = (x_1 \star_1 y_1, \dots, x_n \star_n y_n)$; the element $\mathbf{0}$ is the zero of this semiring.

Note that the set $\tilde{S} = S_1 \times \cdots \times S_n$ is also an idempotent semiring with respect to the same operations; its zero is the element $(\mathbf{0}_1, \dots, \mathbf{0}_n)$.

Notice that even if primitive semirings in examples 3.1 and 3.2 are linearly ordered sets with respect to the orders induced by the correspondent addition operations, the derived semirings are only partially ordered.

EXAMPLE 3.3. Let S be an idempotent semiring and $\text{Mat}_{mn}(S)$ be a set of all S -valued matrices. Define the sum \oplus of matrices $A = (a_{ij}), B = (b_{ij}) \in \text{Mat}_{mn}(S)$ as $A \oplus B = (a_{ij} \oplus b_{ij}) \in \text{Mat}_{mn}(S)$, and let \preceq be the corresponding order on $\text{Mat}_{mn}(S)$. The *product* of two matrices $A \in \text{Mat}_{lm}(S)$ and $B \in \text{Mat}_{mn}(S)$ is a matrix $AB \in \text{Mat}_{ln}(S)$ such that $AB = (\bigoplus_{k=1}^m a_{ik} \odot b_{kj})$. The set $\text{Mat}_{nn}(S)$ of square matrices is an idempotent semiring with respect to these operations. If $\mathbf{0}$ is a zero of S , then the matrix $O = (o_{ij})$, where $o_{ij} = \mathbf{0}$, is the zero of $\text{Mat}_{nn}(S)$; if $\mathbf{1}$ is a unity of S , then

the matrix $E = (\delta_{ij})$, where $\delta_{ij} = \mathbf{1}$ if $i = j$ and $\delta_{ij} = \mathbf{0}$ otherwise, is the unity of $\text{Mat}_{nn}(S)$.

Many additional examples can be found, e.g., in [2]–[8].

4 Idempotency and linearity

Now we discuss an idempotent analog of a linear space. A set V is called a *semimodule* over an idempotent semiring S (or an S -semimodule) if it is equipped with an idempotent commutative associative addition operation \oplus_V and a multiplication $\odot_V: S \times V \rightarrow V$ satisfying the following conditions: for all $\lambda, \mu \in S, v, w \in V$

- $(\lambda \odot \mu) \odot_V v = \lambda \odot_V (\mu \odot_V v)$;
- $\lambda \odot_V (v \oplus_V w) = (\lambda \odot_V v) \oplus_V (\lambda \odot_V w)$;
- $(\lambda \oplus \mu) \odot_V v = (\lambda \odot_V v) \oplus_V (\mu \odot_V v)$.

An S -semimodule V is called a *semimodule with zero* if S is a semiring with a zero $\mathbf{0} \in S$ and there exists a zero element $\mathbf{0}_V \in V$ such that for all $v \in V, \lambda \in S$

- $\mathbf{0}_V \oplus_V v = v$;
- $\lambda \odot_V \mathbf{0}_V = \mathbf{0} \odot_V v = \mathbf{0}_V$.

EXAMPLE 4.1. FINITELY GENERATED FREE SEMIMODULE. The simplest S -semimodule is the direct product $S^n = \{ (a_1, \dots, a_n) \mid a_j \in S, j = 1, \dots, n \}$. The set of all endomorphisms $S^n \rightarrow S^n$ coincides with the semiring $\text{Mat}_{nn}(S)$ of all S -valued matrices (see example 3.3).

EXAMPLE 4.2. MATRIX SEMIMODULE. Take some $c \in S, A \in \text{Mat}_{mn}(S)$. The product $c \odot A$ is defined as the matrix $(c \odot a_{ij}) \in \text{Mat}_{mn}(S)$. Then the set of all S -valued matrices of a given order $\text{Mat}_{mn}(S)$ forms a semimodule under addition \oplus and multiplication by elements of S .

EXAMPLE 4.3. FUNCTION SPACES. An *idempotent function space* $\mathcal{F}(X; S)$ is a subset of the set of all maps $X \rightarrow S$ such that if $f(x), g(x) \in \mathcal{F}(X; S)$ and $c \in S$, then $(f \oplus g)(x) = f(x) \oplus g(x) \in \mathcal{F}(X; S)$ and $(c \odot f)(x) = c \odot f(x) \in \mathcal{F}(X; S)$; thus an idempotent function space is another example of an S -semimodule. If the semiring S contains a zero element $\mathbf{0}$ and $\mathcal{F}(X; S)$ contains the zero constant function $o(x) = \mathbf{0}$, then the function space $\mathcal{F}(X; S)$ has the structure of a semimodule with zero $o(x)$ over the semiring S .

Recall that the idempotent addition defines a partial order in semiring S . An important example of an idempotent functional space is the space $\mathcal{B}(X; S)$ of all functions $X \rightarrow S$ bounded from above with respect to the partial order \preceq in S . There are many interesting spaces of this type including $\mathcal{C}(X; S)$ (a space of continuous functions defined on a topological space X), analogs of the Sobolev spaces, etc. (see, e.g., [2]–[6] for details).

According to the correspondence principle, many important concepts, ideas and results can be converted from usual functional analysis to Idempotent Analysis. For example, an idempotent scalar product in $\mathcal{B}(X; S)$ can be defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x) dx,$$

where the integral is defined as the sup operation (see example 2.1). Notice, however, that in the general case the ordering \preceq in S is not linear.

EXAMPLE 4.4. INTEGRAL OPERATORS. It is natural to construct idempotent analogs of *integral operators* of the form

$$K : \varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x, y) \odot \varphi(y) dy,$$

where $\varphi(y)$ is an element of a functional space $\mathcal{F}_1(Y; S)$, $(K\varphi)(x)$ belongs to a space $\mathcal{F}_2(X; S)$ and $K(x, y)$ is a function $X \times Y \rightarrow S$. Such operators are homomorphisms of the corresponding functional semimodules. If $S = \mathbb{R}_{\max}$, then this definition turns into the formula

$$(K\varphi)(x) = \sup_{y \in Y} (K(x, y) + \varphi(y)).$$

Formulas of this type are standard for optimization problems (see, e.g., [17]).

5 Idempotent superposition principle

In Quantum Mechanics the superposition principle means that the Schrödinger equation (which is basic for the theory) is linear. Similarly in Idempotent Mathematics the (idempotent) superposition principle means that some important and basic problems and equations (e.g., optimization problems, the Bellman equation and its versions and generalizations, the Hamilton-Jacobi equation) nonlinear in the usual sense can be treated as linear over appropriate idempotent semirings, see [1]–[4].

The linearity of the Hamilton-Jacobi equation over \mathbb{R}_{\min} (and \mathbb{R}_{\max}) can be deduced from the usual linearity (over \mathbb{C}) of the corresponding Schrödinger equation by means of the dequantization procedure described above (in section 2). In this case the parameter \hbar of this dequantization coincides with $i\hbar$, where \hbar is the Planck constant; so in this case \hbar must take imaginary values (because $h > 0$; see [5, 6] for details). Of course, this is closely related to variational principles of mechanics; in particular, the Feynman path integral representation of solution to the Schrödinger equation corresponds to the Lax-Oleinik formula for solution of the Hamilton-Jacobi equation (for the latter see, e.g., [18]).

The situation is similar for the differential Bellman equation, see [1, 3].

B.A. Carré [19] used the idempotent linear algebra to show that different optimization problems for finite graphs can be formulated in a unified manner and reduced to

solving Bellman equations, i.e., systems of linear algebraic equations over idempotent semirings.

DISCRETE BELLMAN EQUATION. It is well-known that discrete versions of the Bellman equation can be treated as linear over appropriate idempotent semirings. The following equation (the discrete stationary Bellman equation) plays an important role in both discrete optimization theory and idempotent matrix theory:

$$X = AX \oplus B, \quad (1)$$

where $A \in \text{Mat}_{nn}(S)$, $X, B \in \text{Mat}_{nl}(S)$; matrices A, B are given and X is unknown. Equation (1) is a natural counterpart of the usual linear system $AX = B$ in traditional linear algebra.

Note that if the closure matrix A^* exists, then the matrix $X = A^*B$ satisfies (1) because $A^* = AA^* \oplus E$. It can be easily checked that this special solution is the minimal element of the set of all solutions to (1).

Actually, the theory of the discrete stationary Bellman equation can be developed using the identity $A^* = AA^* \oplus E$ as an additional axiom (the so-called *closed semirings*; see, e.g., [23]).

B. A. Carré [19] also generalized to the idempotent case the principal algorithms of computational linear algebra and showed that some of these coincide with algorithms independently developed for solution of optimization problems; for example, Bellman's method of solving shortest path problems corresponds to a version of Jacobi's method for solving systems of linear equations, whereas Ford's algorithm corresponds to a version of Gauss–Seidel's method.

We stress that these well-known results can be interpreted as a manifestation of the idempotent superposition principle.

6 Idempotent matrix theory: Some results

MATRIX ALGEBRA AND GRAPH THEORY. Any square matrix $A = (a_{ij}) \in \text{Mat}_{nn}(S)$ specifies a weighted directed graph with n nodes such that $a_{ij} \in S$ is the weight of the arc connecting i th node to j th one. On the other hand, let G be a directed graph with at most one arc between any two nodes, every arc of which is characterized by its weight that belongs to S . Then G can be described by a matrix $A \in \text{Mat}_{nn}(S)$; in particular, nonexistent arcs are counted with weight $\mathbf{0}$, and nonzero diagonal entries a_{ii} of matrix A correspond to loops.

For any $A \in \text{Mat}_{nn}(S)$ define $A^0 = E$, $A^k = AA^{k-1}$, $k \geq 1$. Let $a_{ij}^{(k)}$ be (i, j) th element of the matrix A^k ; then it is easy to check that

$$a_{ij}^{(k)} = \bigoplus_{i_0=i, 1 \leq i_1, \dots, i_{k-1} \leq n, i_k=j} a_{i_0 i_1} \odot \dots \odot a_{i_{k-1} i_k}.$$

Consider a *path* of length k in the graph G , i.e. a sequence of nodes i_0, \dots, i_k connected by k arcs with weights $a_{i_0 i_1}, \dots, a_{i_{k-1} i_k}$. The weight of the whole path is defined to be

the product $a_{i_0 i_1} \odot \cdots \odot a_{i_{k-1} i_k}$. Thus $a_{ij}^{(k)}$ is the supremum of weights of all paths of length k connecting node $i_0 = i$ to node $i_k = j$.

ALGEBRAIC PATH PROBLEM. This well-known problem is formulated as follows: for each pair (i, j) calculate supremum of weights of all paths (of arbitrary length) connecting node i to node j . If the semiring S under investigation is \mathbb{R}_{\min} and arc weights are lengths in some metric, then to solve this problem means to find all shortest paths. Is S is $\{\mathbf{0}, \mathbf{1}\}$ and the corresponding directed graph depicts some relation R in the set $\{1, \dots, n\}$, then to solve this problem means to find the transitive closure of R .

It is evident that in terms of matrix theory the algebraic path problem is reduced to finding the matrix satisfying the formal expansion

$$A^* = E \oplus A \oplus A^2 \oplus \cdots = \bigoplus_{k=0}^{\infty} A^k.$$

The matrix A^* is called the *closure* of the matrix A . If the idempotent semiring S is not a -complete, this problem can be nontrivial since we cannot simply take the infinite sum. Below we shall discuss a sufficient condition for the existence of a closure, following work of B. A. Carré [19].

The matrix $A = (a_{ij}) \in \text{Mat}_{nn}(S)$ is said to be *definite* (*semi-definite*) if

$$a_{i_0 i_1} \odot \cdots \odot a_{i_{k-1} i_k} \prec \mathbf{1} \quad (a_{i_0 i_1} \odot \cdots \odot a_{i_{k-1} i_k} \preceq \mathbf{1})$$

for any path (i_1, \dots, i_l) such that $i_0 = i_k$ (i.e., for any closed path). Obviously, every definite matrix is semi-definite. This definition is similar to that of B. A. Carré [19] with the only difference: Carré considers an ordering that is opposite to \preceq .

Theorem 1 (Carré) *Let A be a semi-definite matrix. Then*

$$\bigoplus_{l=0}^k A^l = \bigoplus_{l=0}^{n-1} A^l$$

for $k \geq n - 1$, so the closure matrix $A^* = \bigoplus_{k=0}^{\infty} A^k$ exists and is equal to $\bigoplus_{k=0}^{n-1} A^k$.

For the proof see [19, Theorem 4.1]. The basic idea of the proof is evident: in the graph of a semi-definite matrix it is impossible to construct a path of arbitrarily large weight since the weight of any closed part of a path cannot be greater than $\mathbf{1}$. Thus there exists a universal bound on path weights, which makes possible truncation of the infinite series for the closure matrix.

SPECTRAL THEORY. The spectral theory of matrices whose elements lie in an idempotent semiring is similar to the well-known Perron–Frobenius theory of nonnegative matrices (see, e.g., [3, 8, 15, 16]).

Recall that the matrix $A = (a_{ij}) \in \text{Mat}_{nn}(S)$ is said to be *irreducible* in the sense of [8] if for any $1 \leq i, j \leq n$ either $a_{ij} \neq \mathbf{0}$ or there exist $1 \leq i_1, \dots, i_k \leq n$ such that $a_{ii_1} \odot \cdots \odot a_{i_k j} \neq \mathbf{0}$. In [15, 16] matrices with this property are called indecomposable.

We borrow the following important result from [15, 16] (see also [8]):

Theorem 2 (Dudnikov, Samborskii) *If a commutative idempotent semiring S with a zero $\mathbf{0}$ and a unity $\mathbf{1}$ is algebraically closed and satisfies cancellation and stabilization conditions, then for any matrix $A \in \text{Mat}_{nn}(S)$ there exist a nonzero ‘eigenvector’ $V \in \text{Mat}_{n1}(S)$ and an ‘eigenvalue’ $\lambda \in S$ such that $AV = \lambda \odot V$. If the matrix A is irreducible, then the ‘eigenvalue’ λ is determined uniquely.*

For the proof see [16, Theorem 6.2].

An application of this result will be given in section 12.

Similar results hold for semimodules of bounded or continuous functions [3].

7 Correspondence principle for computations

Of course, the (idempotent) correspondence principle is valid for algorithms as well as for their software and hardware implementations [4, 9]. Thus:

If we have an important and interesting numerical algorithm, then there is a good chance that its semiring analogs are important and interesting as well.

In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. Note that numerical algorithms for standard infinite-dimensional linear problems over idempotent semirings (i.e., for problems related to idempotent integration, integral operators and transformations, the Hamilton-Jacobi and generalized Bellman equations) deal with the corresponding finite-dimensional (or finite) “linear approximations”. Nonlinear algorithms often can be approximated by linear ones. Thus idempotent linear algebra is the basis of the idempotent numerical analysis.

Moreover, it is well-known that algorithms of linear algebra are convenient for parallel computations; their idempotent analogs admit parallelization as well. Thus we obtain a systematic way of applying parallel computation to optimization problems.

Basic algorithms of linear algebra (such as inner product of two vectors, matrix addition and multiplication, etc.) often do not depend on concrete semirings, as well as on the nature of domains containing the elements of vectors and matrices. Thus it seems reasonable to develop *universal algorithms* that can deal equally well with initial data of different domains sharing the same basic structure [4, 9]; an example of such universal Gauss–Jordan elimination algorithm is found in [20].

Numerical algorithms are combinations of basic operations with ‘numbers’, which are elements of some numerical *domains* (e.g., real numbers, integers, etc.). But every computer uses some finite *models* or finite representations of these domains. Discrepancies between ‘ideal’ numbers and their ‘real’ representations lead to calculation errors. This is another reason to deal with universal algorithms that allow to choose a concrete semiring and take into account the effects of its concrete finite representation in a systematic way; see [4, 9] for details and applications of the correspondence principle to hardware and software design.

8 Set-valued idempotent arithmetic

Suppose S is an idempotent semiring and \mathcal{S} is a system of its subsets. We shall denote the elements of \mathcal{S} by $\mathbf{x}, \mathbf{y}, \dots$ and define $\mathbf{x} \star \mathbf{y} = \{t \in S \mid t = x \star y, x \in \mathbf{x}, y \in \mathbf{y}\}$.

We shall suppose that \mathcal{S} satisfies the following two conditions:

- if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and \star is an algebraic operation in S , then there exists $\mathbf{z} \in \mathcal{S}$ such that $\mathbf{z} \supset \mathbf{x} \star \mathbf{y}$;
- if $\{\mathbf{x}_\alpha\}$ is a subset of \mathcal{S} such that $\bigcap_\alpha \mathbf{x}_\alpha \neq \emptyset$, then there exists the infimum of $\{\mathbf{x}_\alpha\}$ in \mathcal{S} with respect to the ordering \subset , i.e., the set $\mathbf{y} \in \mathcal{S}$ such that $\mathbf{y} \subset \bigcap_\alpha \mathbf{x}_\alpha$ and $\mathbf{z} \subset \mathbf{y}$ for any $\mathbf{z} \in \mathcal{S}$ such that $\mathbf{z} \subset \bigcap_\alpha \mathbf{x}_\alpha$.

Define algebraic operations $\overline{\oplus}, \overline{\odot}$ in \mathcal{S} as follows: if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, then $\mathbf{x} \overline{\star} \mathbf{y}$ is the infimum of the set of all elements $\mathbf{z} \in \mathcal{S}$ such that $\mathbf{z} \supset \mathbf{x} \star \mathbf{y}$.

Proposition 1 *The following assertions are true:*

1. \mathcal{S} is closed with respect to operations $\overline{\oplus}, \overline{\odot}$.
2. The element $\mathbf{x} \overline{\star} \mathbf{y}$ is optimal in the following sense: suppose the exact values of input variables x and y lie in sets \mathbf{x} and \mathbf{y} , respectively; then the result of an algebraic operation $\mathbf{x} \overline{\star} \mathbf{y}$ contains the quantity $x \star y$ and is the least subset of S in \mathcal{S} with this property.
3. If the system \mathcal{S} contains all one-element subsets of S , then the semiring $\{S, \oplus, \odot\}$ is isomorphic to a subset of the algebra $\{S, \overline{\oplus}, \overline{\odot}\}$.

The proof is straightforward.

In general, not much can be said about the algebra $\{S, \overline{\oplus}, \overline{\odot}\}$, as the following example shows.

EXAMPLE 8.1. Let $\mathcal{S} = 2^S$ and $\mathbf{x} \overline{\star} \mathbf{y} = \mathbf{x} \star \mathbf{y}$. In general, the set \mathcal{S} with these ‘naïve’ operations $\overline{\oplus}, \overline{\odot}$ satisfies the above assumptions but is not an idempotent semiring. Indeed, let S be the semiring $(\mathbb{R}_{\max} \setminus \{0\})^2 \cup \{(0, 0)\}$ with operations \oplus, \odot defined as in example 3.2. Consider a set $\mathbf{x} = \{(0, 1), (1, 0)\} \in \mathcal{S}$; we see that $\mathbf{x} \overline{\oplus} \mathbf{x} = \{(0, 1), (1, 0), (1, 1)\} \neq \mathbf{x}$ and if $\mathbf{y} = \{(1, 0)\}, \mathbf{z} = \{(0, 1)\}$, then $\mathbf{x} \overline{\odot} (\mathbf{y} \overline{\oplus} \mathbf{z}) = \{(1, 2), (2, 1)\} \neq (\mathbf{x} \overline{\odot} \mathbf{y}) \overline{\oplus} (\mathbf{x} \overline{\odot} \mathbf{z}) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Thus \mathcal{S} with operations $\overline{\oplus}, \overline{\odot}$ does not satisfy axioms of idempotency and distributivity.

It follows that \mathcal{S} should satisfy some additional conditions in order to have the structure of an idempotent semiring. In the next sections we consider the case when \mathcal{S} is a set of all closed intervals; this case is of particular importance since it represents an idempotent analog of the traditional Interval Analysis.

9 Weak interval extension of an idempotent semiring

Let S be an idempotent semiring. A (closed) *interval* in S is a set of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{t \in S \mid \underline{\mathbf{x}} \preceq t \preceq \overline{\mathbf{x}}\}$, where $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in S$ ($\underline{\mathbf{x}} \preceq \overline{\mathbf{x}}$) are said to be *lower* and *upper bounds* of the interval \mathbf{x} , respectively.

Note that if \mathbf{x} and \mathbf{y} are intervals in S , then $\mathbf{x} \subset \mathbf{y}$ iff $\underline{\mathbf{y}} \preceq \underline{\mathbf{x}} \preceq \overline{\mathbf{x}} \preceq \overline{\mathbf{y}}$. In particular, $\mathbf{x} = \mathbf{y}$ iff $\underline{\mathbf{x}} = \underline{\mathbf{y}}$ and $\overline{\mathbf{x}} = \overline{\mathbf{y}}$.

REMARK 9.1. Let \mathbf{x}, \mathbf{y} be intervals in S . In general, the set $\mathbf{x} \star \mathbf{y}$ is not an interval in S . Indeed, consider a set $S = \{\mathbf{0}, a, b, c, d\}$ and let \oplus be defined by the following order relation: $\mathbf{0}$ is the least element, d is the greatest element, and a, b , and c are noncomparable with each other. If \odot is a zero multiplication, i.e., if $x \odot y = \mathbf{0}$ for all $x, y \in S$, then S is an idempotent semiring without unity. Let $\mathbf{x} = [\mathbf{0}, a]$ and $\mathbf{y} = [\mathbf{0}, b]$; thus $\mathbf{x} \oplus \mathbf{y} = \{\mathbf{0}, a, b, d\}$. This set is not an interval since it does not contain c although $\mathbf{0} \preceq c \preceq d$.

Let S be an idempotent semiring. A *weak interval extension* $I(S)$ of the semiring S is the set of all intervals in S equipped with the operations $\overline{\oplus}, \overline{\odot}$, where for any two intervals $\mathbf{x}, \mathbf{y} \in I(S)$ $\mathbf{x} \overline{\star} \mathbf{y}$ is defined as the smallest interval containing the set $\mathbf{x} \star \mathbf{y}$.

Proposition 2 *$I(S)$ is closed with respect to the operations $\overline{\oplus}, \overline{\odot}$ and has the structure of an idempotent semiring. Moreover, $\mathbf{x} \overline{\star} \mathbf{y} = [\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$ for all $\mathbf{x}, \mathbf{y} \in I(S)$.*

PROOF. Take $t \in \mathbf{x} \star \mathbf{y}$ and let $x \in \mathbf{x}, y \in \mathbf{y}$ be such that $t = x \star y$. By definition of interval, it follows that $\underline{\mathbf{x}} \preceq x \preceq \overline{\mathbf{x}}$ and $\underline{\mathbf{y}} \preceq y \preceq \overline{\mathbf{y}}$. Since the operation \star is consistent with the order \preceq , we see that $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preceq x \star y \preceq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$. In particular, $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preceq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$, i.e., the interval $[\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$ is well defined. It follows that $\mathbf{x} \star \mathbf{y} \subset [\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$.

Now let an interval \mathbf{z} in S be such that $\mathbf{x} \star \mathbf{y} \subset \mathbf{z}$. We have $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \in \mathbf{x} \star \mathbf{y} \subset \mathbf{z}$ and $\overline{\mathbf{x}} \star \overline{\mathbf{y}} \in \mathbf{x} \star \mathbf{y} \subset \mathbf{z}$; hence $\underline{\mathbf{z}} \preceq \underline{\mathbf{x}} \star \underline{\mathbf{y}}$ and $\overline{\mathbf{x}} \star \overline{\mathbf{y}} \preceq \overline{\mathbf{z}}$. Since $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preceq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$ by the above, it follows that $[\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}] \subset \mathbf{z}$.

Thus the set $I(S)$ with the operations $\overline{\oplus}, \overline{\odot}$ can be identified with a subset of an idempotent semiring $S \times S$ (see example 3.2). Since $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preceq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$ whenever $\underline{\mathbf{x}} \preceq \overline{\mathbf{x}}$ and $\underline{\mathbf{y}} \preceq \overline{\mathbf{y}}$, $I(S)$ is closed with respect to $\overline{\oplus}, \overline{\odot}$; hence it is an idempotent semiring (a subsemiring of $S \times S$). \square

REMARK 9.2. Note that $I(S)$ satisfies the third condition of proposition 1 only if the semiring S is b -complete; nevertheless, the operations $\overline{\oplus}, \overline{\odot}$ are well defined in the general case.

Proposition 3 *If S is an a -complete (b -complete) idempotent semiring, then $I(S)$ is an a -complete (b -complete) idempotent semiring.*

PROOF. Let S be an a -complete (b -complete) idempotent semiring and $\{\mathbf{x}_\alpha\}$ be a nonempty subset of $I(S)$ (a nonempty subset of $I(S)$ such that the set $\{\overline{\mathbf{x}}_\alpha\}$ is

bounded in S). We claim that the interval $[\bigoplus_{\alpha} \underline{x}_{\alpha}, \bigoplus_{\alpha} \overline{x}_{\alpha}]$ contains the set $N = \{t \in S \mid (\forall \alpha)(\exists x_{\alpha} \in \underline{x}_{\alpha}) t = \bigoplus_{\alpha} x_{\alpha}\}$ and is contained in every other interval containing N ; hence $\bigoplus_{\alpha} \underline{x}_{\alpha} = [\bigoplus_{\alpha} \underline{x}_{\alpha}, \bigoplus_{\alpha} \overline{x}_{\alpha}]$. Indeed, in the case of an a -complete semiring this statement is proved similarly to proposition 2. If S is b -complete and the set $\{\overline{x}_{\alpha}\}$ is bounded from above, then the set $\{\underline{x}_{\alpha}\}$ is also bounded from above, i.e., there exists $y \in S$ such that $\underline{x}_{\alpha} \preceq \overline{x}_{\alpha} \preceq y$ for all α . Thus there exist $\bigoplus_{\alpha} \underline{x}_{\alpha} \in S$ and $\bigoplus_{\alpha} \overline{x}_{\alpha} \in S$. Now the obvious adaptation of the above proof completes the argument.

If S is a -complete and $X \subset I(S)$ is empty, then by above definitions $\overline{\bigoplus} X = [\mathbf{0}, \mathbf{0}]$ and for all $\mathbf{y} \in I(S)$ $\mathbf{y} \odot (\overline{\bigoplus} X) = (\overline{\bigoplus} X) \odot \mathbf{y} = [\mathbf{0}, \mathbf{0}]$. A direct calculation shows that if $X = \{\underline{x}_{\alpha}\} \subset I(S)$ is nonempty, then

$$\mathbf{y} \odot \left(\overline{\bigoplus_{\alpha} \underline{x}_{\alpha}} \right) = \overline{\bigoplus_{\alpha}} (\mathbf{y} \odot \underline{x}_{\alpha}), \quad \left(\overline{\bigoplus_{\alpha} \underline{x}_{\alpha}} \right) \odot \mathbf{y} = \overline{\bigoplus_{\alpha}} (\underline{x}_{\alpha} \odot \mathbf{y})$$

for any $\mathbf{y} \in I(S)$. Thus $I(S)$ is a -complete (b -complete) if S is a -complete (b -complete). \square

The following two propositions are straightforward consequences of proposition 2:

Proposition 4 *If an idempotent semiring S is commutative, so is $I(S)$.*

Proposition 5 *If an idempotent semiring S contains a zero $\mathbf{0}$ (unity $\mathbf{1}$), then the interval $[\mathbf{0}, \mathbf{0}]$ ($[\mathbf{1}, \mathbf{1}]$) is the zero (unity) of $I(S)$.*

Proposition 6 *If an idempotent semiring S has a zero $\mathbf{0}$ and does not contain zero divisors, $I(S)$ has no zero divisors as well.*

PROOF. Let $\underline{x}, \underline{y} \in I(S)$ and $\underline{x} \neq [\mathbf{0}, \mathbf{0}]$, $\underline{y} \neq [\mathbf{0}, \mathbf{0}]$. Because $\underline{x} \preceq \overline{x}$, $\underline{y} \preceq \overline{y}$, this means that $\overline{x} \neq \mathbf{0}$, $\overline{y} \neq \mathbf{0}$. Thus if $\underline{z} = \underline{x} \odot \underline{y}$, then $\overline{z} = \overline{x} \odot \overline{y} \neq \mathbf{0}$, since there are no zero divisors in S . It follows that $\underline{z} \neq [\mathbf{0}, \mathbf{0}]$. \square

Proposition 7 *If S is algebraically closed and for any $x, y \in S$, $n \in \mathbb{N}$ the equality $(x \oplus y)^n = x^n \oplus y^n$ holds, then $I(S)$ is algebraically closed.*

PROOF. Suppose $\underline{x}^n = \underline{x} \odot \underline{x} \odot \cdots \odot \underline{x} = \underline{y}$. By proposition 2, we see that $\underline{x}^n = \underline{y}$ and $\overline{x}^n = \overline{y}$. Let $\underline{z} \in S$ and $\overline{z} \in S$ be the solutions of these two equations. We claim that \underline{z} and \overline{z} can be chosen such that $\underline{z} \preceq \overline{z}$, i.e., the interval $[\underline{z}, \overline{z}]$ is well defined.

Take $\overline{z}' = \underline{z} \oplus \overline{z}$; hence $\underline{z} \preceq \overline{z}'$. Since in S $\overline{z}^n = (\underline{z} \oplus \overline{z})^n = \underline{z}^n \oplus \overline{z}^n$, $\overline{z}^n = \underline{y} \oplus \overline{y} = \overline{y}$. We see that $[\underline{z}, \overline{z}']^n = [\underline{y}, \overline{y}] = \underline{y}$. \square

REMARK 9.3. Note that the equality $(x \oplus y)^n = x^n \oplus y^n$ holds in any commutative idempotent semiring S satisfying cancellation condition (see, e.g., [16], assertion 2.1).

10 A stronger notion of interval extension

We stress that in general a weak interval extension $I(S)$ of an idempotent semiring S that satisfies cancellation and stabilization conditions does not inherit the latter two properties. Indeed, let $\mathbf{x} \odot \mathbf{z} = \mathbf{y} \odot \mathbf{z}$, where $\mathbf{z} = [\mathbf{0}, z]$ and $z \neq \mathbf{0}$; then \mathbf{z} is a nonzero element but this does not imply that $\mathbf{x} = \mathbf{y}$ since $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ may not equal each other. Further, let $\mathbf{y} = [\mathbf{0}, y] \neq [\mathbf{0}, \mathbf{0}]$; then the lower bound of $\mathbf{x}^n \odot \mathbf{y}$ may not stabilize when $n \rightarrow \infty$.

Therefore we define a stronger notion of *interval extension* of an idempotent semiring S with a zero $\mathbf{0}$ to be the set $\mathbf{I}(S) = \{ [x, y] \mid x, y \in S, \mathbf{0} \prec x \preceq y, \} \cup \{[\mathbf{0}, \mathbf{0}]\}$ equipped with operations \oplus, \odot defined as above.

Note that this object may not be well-defined if the semiring S has zero divisors. Indeed, let $\mathbf{x} = [x_1, x_2] \in \mathbf{I}(S)$ and $\mathbf{y} = [y_1, y_2] \in \mathbf{I}(S)$ be such that $\mathbf{0} \prec x_1 \prec x_2$, $\mathbf{0} \prec y_1 \prec y_2$, $x_1 \odot y_1 = \mathbf{0}$, and $x_2 \odot y_2 \neq \mathbf{0}$; then $\mathbf{x} \odot \mathbf{y} = [\mathbf{0}, x_2 \odot y_2] \notin \mathbf{I}(S)$.

Throughout this section, we will suppose the interval extension $\mathbf{I}(S)$ of an idempotent semiring S to be closed with respect to the operations \oplus and \odot . To achieve this, it is sufficient to require that the semiring S contains no zero divisors.

Theorem 3 *The set $\mathbf{I}(S)$ is an idempotent semiring with respect to the operations \oplus and \odot with the zero $[\mathbf{0}, \mathbf{0}]$ and does not contain zero divisors. It inherits some special properties of the semiring S :*

1. *If S is a-complete (b-complete), then $\mathbf{I}(S)$ is a-complete (b-complete).*
2. *If S is commutative, so is $\mathbf{I}(S)$.*
3. *If $\mathbf{1}$ is a unity of S , $[\mathbf{1}, \mathbf{1}]$ is the unity of $\mathbf{I}(S)$.*
4. *If S is algebraically closed and for any $x, y \in S$, $n \in \mathbb{N}$ the equality $(x \oplus y)^n = x^n \oplus y^n$ holds, then $\mathbf{I}(S)$ is algebraically closed.*
5. *If S satisfies the cancellation condition, so does $\mathbf{I}(S)$.*
6. *If S satisfies the stabilization condition, so does $\mathbf{I}(S)$.*

PROOF. Using proposition 2, it is easy to check that $\mathbf{I}(S)$ is an idempotent semiring with respect to the operations \oplus, \odot . This semiring has the zero element $[\mathbf{0}, \mathbf{0}]$ but contains no zero divisors by propositions 5 and 6. Propositions 3–5 and 7 imply the first four statements.

Suppose S satisfies the cancellation condition, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{I}(S)$, and \mathbf{z} is nonzero. If $\mathbf{x} \odot \mathbf{z} = \mathbf{y} \odot \mathbf{z}$, then $\underline{\mathbf{x}} \odot \underline{\mathbf{z}} = \underline{\mathbf{y}} \odot \underline{\mathbf{z}}$ and $\overline{\mathbf{x}} \odot \overline{\mathbf{z}} = \overline{\mathbf{y}} \odot \overline{\mathbf{z}}$; since $\mathbf{z} \neq [\mathbf{0}, \mathbf{0}]$ in $\mathbf{I}(S)$, $\underline{\mathbf{z}} \neq \mathbf{0}$ and $\overline{\mathbf{z}} \neq \mathbf{0}$, and it follows from the assumptions that $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = [\underline{\mathbf{y}}, \overline{\mathbf{y}}] = \mathbf{y}$. If $\mathbf{z} \odot \mathbf{x} = \mathbf{z} \odot \mathbf{y}$, then $\mathbf{x} = \mathbf{y}$ similarly.

Suppose further that S satisfies the stabilization condition; by definition, $\underline{\mathbf{y}} \neq \mathbf{0}$ and $\overline{\mathbf{y}} \neq \mathbf{0}$ for any nonzero $\mathbf{y} \in \mathbf{I}(S)$. Consider the sequence $\mathbf{x}^n \oplus \mathbf{y}$; stabilization

holds in S for both bounds of the involved intervals and hence, by proposition 2, for the whole intervals as elements of $\mathbf{I}(S)$. \square

Suppose S is an idempotent semiring; then the map $\iota: S \rightarrow I(S)$ defined by $\iota(x) = [x, x]$ is an isomorphic imbedding of S into its weak interval extension $I(S)$. If the semiring S has a zero $\mathbf{0}$ but no zero divisors, then the map $\iota: S \rightarrow \mathbf{I}(S) \subset I(S)$ is an isomorphic imbedding of S into its interval extension. In the sequel, we will identify the semiring S with subsemirings $\iota(S) \subset I(S)$ or $\iota(S) \subset \mathbf{I}(S) \subset I(S)$ and denote the operations in $I(S)$ or $\mathbf{I}(S)$ by \oplus, \odot . If the semiring S contains a unity $\mathbf{1}$, then we denote the unit element $[\mathbf{1}, \mathbf{1}]$ of $I(S)$ or $\mathbf{I}(S)$ by $\mathbf{1}$; similarly, we denote $[\mathbf{0}, \mathbf{0}]$ by $\mathbf{0}$.

11 Cancellation and semifields

We stress that in idempotent interval mathematics most of algebraic properties of an idempotent semiring are conserved in its interval extension. On the contrary, if S is an idempotent semifield, then the set $\mathbf{I}(S)$ is not a semifield but only a semiring satisfying the cancellation condition.

Recall that any commutative idempotent semiring S with a zero $\mathbf{0}$ can be isomorphically embedded into an idempotent semifield \tilde{S} provided that S satisfies the cancellation condition (see, e.g., [15]). If \tilde{S} coincides with its subsemifield generated by S , then \tilde{S} is called a *semifield of fractions* corresponding to the semiring S . This semifield can be constructed as the quotient $S \times (S \setminus \{\mathbf{0}\}) / \sim$, where for any $(x, y), (z, t) \in S \times (S \setminus \{\mathbf{0}\})$

$$(x, y) \sim (z, t) \iff x \odot t = y \odot z,$$

equipped with operations

$$(x, y) \oplus (z, t) = ((x \odot t) \oplus (y \odot z), y \odot t), \quad (x, y) \odot (z, t) = (x \odot z, y \odot t).$$

It is easy to see that these operations are defined in such a way that pairs (x, y) are treated as ‘fractions’ with x as ‘numerator’ and y as (nonzero) ‘denominator’. This semifield has the zero element $\{(\mathbf{0}, y) \mid y \neq \mathbf{0}\}$ and the unit element $\{(y, y) \mid y \neq \mathbf{0}\}$; for every ‘fraction’ (x, y) representing a nonzero element of \tilde{S} its inverse element is given by the fraction (y, x) .

In the context of the traditional Interval Analysis a similar extension of the algebra of numerical intervals leads to the so-called *Kaucher interval arithmetic* [21, 22]. In addition to usual intervals $[x, y]$, where $x \leq y$, it includes quasiintervals $[x, y]$ with $y \leq x$, which arise as inverse elements for the former.

Here we describe an idempotent version of Kaucher arithmetic. The following statement shows that in this case the semifield of fractions of interval extension $\mathbf{I}(S)$ corresponding to an idempotent semiring S with cancellation condition has very simple structure: it is isomorphic to the idempotent semifield $(\tilde{S} \setminus \{\mathbf{0}\})^2 \cup \{\mathbf{0}\}$.

Proposition 8 *Suppose S is a commutative idempotent semiring with a zero $\mathbf{0}$, S satisfies the cancellation condition, and \tilde{S} is its semifield of fractions; then a semifield*

of fractions corresponding to the interval extension $\mathbf{I}(S)$ is isomorphic to the semifield $(\tilde{S} \setminus \{\mathbf{0}\})^2 \cup \{\mathbf{0}\}$ (see example 3.2).

PROOF. It follows from theorem 3 that $\mathbf{I}(S)$ is a commutative idempotent semiring with the zero element $\mathbf{0} = [\mathbf{0}, \mathbf{0}]$ and satisfies the cancellation condition. Thus $\mathbf{I}(S)$ can be isomorphically embedded into its semifield of fractions.

Define the map $\phi: \mathbf{I}(S) \times (\mathbf{I}(S) \setminus \{\mathbf{0}\}) \rightarrow (\tilde{S} \setminus \{\mathbf{0}\})^2 \cup \{\mathbf{0}\}$ by the rule $\phi((\mathbf{x}, \mathbf{y})) = (\underline{\mathbf{x}} \odot \underline{\mathbf{y}}^{-1}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}^{-1})$. This map is surjective. Indeed, $(\mathbf{0}, \mathbf{0}) = \phi((\mathbf{0}, \mathbf{y}))$ for any $\mathbf{y} \neq \mathbf{0}$; let us check that if $a, b \in \tilde{S}$, $a \neq \mathbf{0}$, $b \neq \mathbf{0}$, then there exist $\mathbf{x}, \mathbf{y} \in \mathbf{I}(S)$, $\mathbf{y} \neq \mathbf{0}$, such that $(a, b) = \phi((\mathbf{x}, \mathbf{y}))$. We see that there exist $a_1, a_2, b_1, b_2 \in S$ such that $a = a_1 \odot a_2^{-1}$, $b = b_1 \odot b_2^{-1}$. Define $\underline{\mathbf{x}} = a_1 \odot b_1 \odot b_2$, $\overline{\mathbf{x}} = (a_1 \odot b_1 \odot b_2) \oplus (a_2 \odot b_1^2)$, $\underline{\mathbf{y}} = a_2 \odot b_1 \odot b_2$, $\overline{\mathbf{y}} = (a_1 \odot b_2^2) \oplus (a_2 \odot b_1 \odot b_2)$; thus $\underline{\mathbf{x}} \preceq \overline{\mathbf{x}}$, $\underline{\mathbf{y}} \preceq \overline{\mathbf{y}}$ and $\underline{\mathbf{x}} \odot \underline{\mathbf{y}}^{-1} = a_1 \odot a_2^{-1} = a$, $\overline{\mathbf{x}} \odot \overline{\mathbf{y}}^{-1} = b_1 \odot b_2^{-1} = b$.

Since $x \odot y^{-1} = z \odot t^{-1}$ iff $x \odot t = y \odot z$ for any $x, y, z, t \in \tilde{S}$, we see that $\phi((\mathbf{x}, \mathbf{y})) = \phi((\mathbf{z}, \mathbf{t}))$ iff (\mathbf{x}, \mathbf{y}) and (\mathbf{z}, \mathbf{t}) define the same element of the semifield of fractions corresponding to $\mathbf{I}(S)$. Also,

$$\begin{aligned} \phi(((\mathbf{x} \odot \mathbf{t}) \oplus (\mathbf{y} \odot \mathbf{z}), \mathbf{y} \odot \mathbf{t})) &= ((\underline{\mathbf{x}} \odot \underline{\mathbf{y}}^{-1}) \oplus (\underline{\mathbf{z}} \odot \underline{\mathbf{t}}^{-1}), (\overline{\mathbf{x}} \odot \overline{\mathbf{y}}^{-1}) \oplus (\overline{\mathbf{z}} \odot \overline{\mathbf{t}}^{-1})) \\ &= \phi((\mathbf{x}, \mathbf{y})) \oplus \phi((\mathbf{z}, \mathbf{t})), \\ \phi((\mathbf{x} \odot \mathbf{z}, \mathbf{y} \odot \mathbf{t})) &= ((\underline{\mathbf{x}} \odot \underline{\mathbf{y}}^{-1}) \odot (\underline{\mathbf{z}} \odot \underline{\mathbf{t}}^{-1}), (\overline{\mathbf{x}} \odot \overline{\mathbf{y}}^{-1}) \odot (\overline{\mathbf{z}} \odot \overline{\mathbf{t}}^{-1})) \\ &= \phi((\mathbf{x}, \mathbf{y})) \odot \phi((\mathbf{z}, \mathbf{t})). \end{aligned}$$

Thus the semifield of fractions corresponding to $\mathbf{I}(S)$ is isomorphic to the idempotent semifield $(\tilde{S} \setminus \{\mathbf{0}\})^2 \cup \{\mathbf{0}\}$. \square

The commutativity condition in this proposition is a natural one. Indeed, it can be proved that if each nonzero element of a b -complete idempotent semigroup S has a multiplicative inverse, then S is commutative (see, e.g., [5]).

12 Application to linear algebra

Suppose S is an idempotent semiring with a zero $\mathbf{0}$ and a unity $\mathbf{1}$ and $I(S)$ is its weak interval extension; then $\text{Mat}_{nn}(I(S))$ is an idempotent semiring. If the interval extension $\mathbf{I}(S)$ of the semiring S is well defined, then the same is true for $\text{Mat}_{nn}(\mathbf{I}(S))$. We shall denote the (common) unit element of these semirings by \mathbf{E} .

If $\mathbf{A} = (\mathbf{a}_{ij}) \in \text{Mat}_{nn}(I(S))$ ($\mathbf{A} = (\mathbf{a}_{ij}) \in \text{Mat}_{nn}(\mathbf{I}(S))$) is a (not necessarily square) interval matrix, then the matrices $\underline{\mathbf{A}} = (\underline{\mathbf{a}}_{ij})$ and $\overline{\mathbf{A}} = (\overline{\mathbf{a}}_{ij})$ are called *lower* and *upper matrices* of the *interval matrix* \mathbf{A} .

Proposition 9 *Let S be an idempotent semiring with a zero $\mathbf{0}$ and a unity $\mathbf{1}$. The mapping $\mathbf{A} \in \text{Mat}_{nn}(I(S)) \mapsto [\underline{\mathbf{A}}, \overline{\mathbf{A}}] \in I(\text{Mat}_{nn}(S))$ is an isomorphism of idempotent semirings $\text{Mat}_{nn}(I(S))$ and $I(\text{Mat}_{nn}(S))$. If the semiring S has an interval extension $\mathbf{I}(S)$, then this assertion remains true if $I(S)$ is substituted by $\mathbf{I}(S)$.*

Here intervals $[\underline{\mathbf{A}}, \overline{\mathbf{A}}]$ in $I(\text{Mat}_{nn}(S))$ or $\mathbf{I}(\text{Mat}_{nn}(S))$ are defined with respect to the partial ordering \preceq in $\text{Mat}_{nn}(S)$ (see example 3.3). The proof follows easily from proposition 2; indeed, this proposition implies that addition and multiplication of interval matrices are reduced to separate addition and multiplication of their lower and upper matrices.

The following proposition is an immediate consequence of theorem 2:

Proposition 10 *If a commutative idempotent semiring S with a zero $\mathbf{0}$ and a unity $\mathbf{1}$ is algebraically closed and satisfies cancellation and stabilization conditions, then for any matrix $\mathbf{A} \in \text{Mat}_{nn}(\mathbf{I}(S))$ there exist a nonzero ‘eigenvector’ $\mathbf{V} \in \text{Mat}_{n1}(\mathbf{I}(S))$ and an ‘eigenvalue’ $[\underline{\lambda}, \overline{\lambda}] \in \mathbf{I}(S)$ such that $\mathbf{A}\mathbf{V} = [\underline{\lambda}, \overline{\lambda}] \odot \mathbf{V}$. If the matrix \mathbf{A} is irreducible, then the ‘eigenvalue’ $[\underline{\lambda}, \overline{\lambda}]$ is determined uniquely.*

It follows from proposition 2 that $\underline{\mathbf{A}}\underline{\mathbf{V}} = \underline{\lambda} \odot \underline{\mathbf{V}}$ and $\overline{\mathbf{A}}\overline{\mathbf{V}} = \overline{\lambda} \odot \overline{\mathbf{V}}$.

Consider the following interval discrete stationary Bellman equation (see also sections 5, refs:matrices):

$$\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{B},$$

where $\mathbf{A} \in \text{Mat}_{nn}(\mathbf{I}(S))$, $\mathbf{B}, \mathbf{X} \in \text{Mat}_{ns}(\mathbf{I}(S))$. Consider the following iterative process:

$$\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k \oplus \mathbf{B} = \mathbf{A}^{k+1}\mathbf{X}_0 \oplus \left(\bigoplus_{l=0}^k \mathbf{A}^l \right) \mathbf{B}, \quad (2)$$

where $\mathbf{X}_k \in \text{Mat}_{ns}(\mathbf{I}(S))$, $k = 0, 1, \dots$

The following proposition is due to B. A. Carré [19, Theorem 6.1] up to some terminology:

Proposition 11 *If matrix $A \in \text{Mat}_{nn}(S)$ is semi-definite, then the iterative process $X_{k+1} = AX_k \oplus B$ stabilizes to the solution $X = A^*B$ of the equation $X = AX \oplus B$ after at most n iterations for any initial approximation $X_0 \in \text{Mat}_{n1}(S)$ such that $X_0 \preceq A^*B$.*

Suppose an idempotent semiring S satisfies the assumptions of proposition 10. Let $[\underline{\lambda}_1, \overline{\lambda}_1], \dots, [\underline{\lambda}_k, \overline{\lambda}_k]$, $1 \leq k \leq n$, be the eigenvalues of the matrix $\mathbf{A} \in \text{Mat}_{nn}(\mathbf{I}(S))$. Denote $\sup\{\underline{\lambda}_1, \dots, \underline{\lambda}_k\} = \bigoplus_{l=1}^k \underline{\lambda}_l$ by $\rho(\mathbf{A})$. It is possible to give a simple spectral criterion of convergence of the iterative process (2):

Theorem 4 *Let S be a commutative semiring satisfying conditions of proposition 10 and matrix $\mathbf{A} \in \text{Mat}_{nn}(\mathbf{I}(S))$ be such that $\rho(\mathbf{A}) \preceq \mathbf{1}$. Then the iterative process $\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k \oplus \mathbf{B}$, $k \geq 0$, stabilizes to the minimal solution $\mathbf{X} = \mathbf{A}^*\mathbf{B}$ of equation $\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{B}$ after at most n iterations for any $\mathbf{X}_0 \in \text{Mat}_{n1}(\mathbf{I}(S))$ such that $\mathbf{X}_0 \preceq \mathbf{X}$.*

PROOF. It follows from proposition 2 that it is sufficient to prove that sequences of lower and upper matrices of $\{\mathbf{X}_k\}$ converge separately. To this end, we shall show that the matrices $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$ are semi-definite; then the result will follow from proposition 11.

Since $\underline{\mathbf{a}}_{ij} \preceq \overline{\mathbf{a}}_{ij}$ for all i, j , we need only to prove that $\overline{\mathbf{A}}$ is semi-definite. First we shall prove this if $\overline{\mathbf{A}}$ is irreducible. Using the expression for a unique eigenvalue of an irreducible matrix $\overline{\mathbf{A}}$ in terms of cycle invariants [16]

$$\overline{\lambda}^{\phi(n)} = \bigoplus_{\substack{l=1, \dots, n \\ (i_1, \dots, i_l)}} [\overline{\mathbf{a}}_{i_1 i_2} \odot \dots \odot \overline{\mathbf{a}}_{i_l i_1}]^{\phi(n)/l},$$

where $\phi(n)$ is the least common multiple of the numbers $1, \dots, n$, we see that for any cycle its cycle invariant $P = \overline{\mathbf{a}}_{i_1 i_2} \odot \dots \odot \overline{\mathbf{a}}_{i_l i_1}$ satisfies $P \preceq \mathbf{1}$ if $\overline{\lambda} \preceq \mathbf{1}$ (indeed, if $P \oplus \mathbf{1} \succ \mathbf{1}$, then, by remark 9.3, $(\mathbf{1} \oplus P)^{\phi(n)/l} = \mathbf{1} \oplus P^{\phi(n)/l} \succ \mathbf{1}$, so $\mathbf{1} \oplus \overline{\lambda}^{\phi(n)} \succ \mathbf{1}$ —a contradiction). Thus $\overline{\mathbf{A}}$ is a semi-definite matrix.

If $\overline{\mathbf{A}}$ is reducible, there exists a permutation matrix Q such that $\overline{\mathbf{A}} = QBQ^{-1}$, where $B = (b_{ij})_{i,j=1, \dots, n}$ has an upper block triangular form:

$$B = \begin{pmatrix} B_1 & * & \dots & * \\ \mathbf{0} & B_2 & \dots & * \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & B_k \end{pmatrix}, \quad 1 < k \leq n,$$

and all square matrices B_1, \dots, B_k are either zero or irreducible. We claim that every eigenvalue $\overline{\lambda}$ of $\overline{\mathbf{A}}$ is an eigenvalue of some B_l , $l = 1, \dots, k$. Indeed, let V be an eigenvector of $\overline{\mathbf{A}}$ with an eigenvalue $\overline{\lambda}$; denote i th element of the vector V by v_i . Consider a decomposition $\{1, \dots, n\} = N_1 \cup \dots \cup N_k$, where $N_i \cap N_j = \emptyset$ if $i \neq j$ and $B_l = (b_{ij})_{i,j \in N_l}$, $l = 1, \dots, k$; let $l_0 = \max\{l \mid \exists i \in N_l: v_i \neq \mathbf{0}\}$. We see that λ is a unique eigenvalue of the irreducible matrix B_{l_0} corresponding to the eigenvector $(v_i)_{i \in N_{l_0}}$. The condition $\rho(\mathbf{A}) \preceq \mathbf{1}$ implies that B_1, \dots, B_k are semi-definite. Since $P = \mathbf{0} \prec \mathbf{1}$ for any cycle containing indices $i \in N_l$, $j \in N_s$, $l \neq s$, we conclude that $\overline{\mathbf{A}}$ is a semi-definite matrix. \square

REMARK 12.1. Compare this simple proposition with the well-known spectral convergence criterion of the iterative process in traditional Interval Analysis ([12, theorem 12.1]), which in our notation has the following form:

The iterative process $\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k + \mathbf{B}$, $k \geq 0$, converges to a unique solution \mathbf{X} of the equation $\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}$ for any $\mathbf{X}_0 \in \text{Mat}_{n1}(I(\mathbb{C}))$ if and only if $\rho(|\mathbf{A}|) < 1$.

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